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Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background

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ABSTRACT

Cosmology's standard model posits an infinite flat universe forever expanding under the pressure of dark energy. First-year data from the Wilkinson Microwave Anisotropy Probe (WMAP) confirm this model to spectacular precision on all but the largest scales (Bennett *et al.*, 2003 ; Spergel *et al.*, 2003). Temperature correlations across the microwave sky match expectations on scales narrower than 60°, yet vanish on scales wider than 60°. Researchers are now seeking an explanation of the missing wide-angle correlations (Contaldi *et al.*, 2003 ; Cline *et al.*, 2003). One natural approach questions the underlying geometry of space, namely its curvature (Efstathiou, 2003) and its topology (Tegmark *et al.*, 2003). In an infinite flat space, waves from the big bang would fill the universe on all length scales. The observed lack of temperature correlations on scales beyond 60° means the broadest waves are missing, perhaps because space itself is not big enough to support them.

Here we present a simple geometrical model of a finite, positively curved space – the Poincaré dodecahedral space – which accounts for WMAP's observations with no fine-tuning required. Circle searching (Cornish, Spergel and Starkman, 1998) may confirm the model's topological predictions, while upcoming Planck Surveyor data may confirm its predicted density of $\Omega_0 \simeq 1.013 > 1$. If confirmed, the model will answer the ancient question of whether space is finite or infinite, while retaining the standard Friedmann-Lemaître foundation for local physics.

Subject headings: cosmology: theory, large scale structure: topology

1. CMB Data Suggest Spherical Space with Dodecahedral Topology

Temperature fluctuations on the microwave sky may be expressed as a sum of spherical harmonics, just as music and other sounds may be expressed as a sum of ordinary harmonics. A musical note is the sum of a fundamental, a second harmonic, a third harmonic, and so on. The relative strengths of the harmonics – the note's spectrum – determines the tone quality, distinguishing, say, a sustained middle C played on a flute from the same note played on a clarinet. Analogously, the temperature map on the microwave sky is the sum of spherical harmonics. The relative strengths of the harmonics – the power spectrum – is a signature of the physics and geometry of the universe. Indeed the power spectrum is the primary tool researchers use to test their models' predictions against observed reality.

The infinite universe model gets into trouble at the low end of the power spectrum (Figure 1). The lowest harmonic – the dipole, with wave number $\ell = 1$ – is unobservable because the Doppler effect of the solar system's motion through space creates a dipole a hundred times stronger, swamping out the underlying cosmological dipole. The first observable harmonic is the quadrupole, with wave number $\ell = 2$. WMAP found a quadrupole only about 1/7 as strong as would be expected in an infinite flat space. The probability that this could happen by mere chance has been estimated at about a fifth of one percent (Spergel *et al.*, 2003). The octopole term, with wave number $\ell = 3$, is also weak at 72% of the expected value, but not nearly so dramatic or significant as the quadrupole. For large values of ℓ, ranging up to $\ell = 900$ and corresponding to small-scale temperature fluctuations, the spectrum tracks the infinite universe predictions exceedingly well.

Cosmologists thus face the challenge of finding a model that accounts for the weak quadrupole while maintaining the success of the infinite flat universe model on small scales (high ℓ). The weak wide-angle temperature correlations discussed in the introductory paragraph correspond directly to the weak quadrupole.

Microwave background temperature fluctuations arise primarily (but not exclusively) from density fluctuations in the early universe, because photons travelling from denser regions do a little extra work against gravity and therefore arrive cooler, while photons from less dense regions do less work against gravity and arrive warmer. The density fluctuations across space split into a sum of 3-dimensional harmonics – in effect the vibrational overtones of space itself – just as temperature fluctuations on the sky split into a sum of 2-dimensional spherical harmonics and a musical note splits into a sum of 1-dimensional harmonics. The low quadrupole implies a cut-off on the wavelengths of the 3-dimensional harmonics. Such a cut-off presents an awkward problem in infinite flat space, because it defines a preferred length scale in an otherwise scale-invariant space. A more natural explanation invokes a finite universe, where the size of space itself imposes a cut-off on the wavelengths (Figure 2). Just as the vibrations of a bell cannot be larger than the bell itself, the density fluctuations in space cannot be larger than space itself. While most potential spatial topologies fail to fit the WMAP results, the Poincaré dodecahedral space fits them strikingly well.

The Poincaré dodecahedral space is a dodecahedral block of space with opposite faces abstractly glued together, so objects passing out of the dodecahedron across any face return from the opposite face. Light travels across the faces in the same way, so if we sit inside the dodecahedron and look outward across a face, our line of sight re-enters the dodecahedron from the opposite face. We have the illusion of looking into an adjacent copy of the dodecahedron. If we take the original dodecahedral block of space not as a Euclidean dodecahedron (with edge angles $\simeq 117°$) but as a spherical dodecahedron (with edge angles exactly $120°$), then adjacent images of the dodecahedron fit together snugly to tile the hypersphere (Figure 3b), analogously to the way adjacent images of spherical pentagons (with perfect 120° angles) fit snugly to tile an ordinary sphere (Figure 3a). Thus the Poincaré space is a positively curved space, with a multiply connected topology whose volume is 120 times smaller than that of the simply connected hypersphere (for a given curvature radius).

The Poincaré dodecahedral space's power spectrum depends strongly on the assumed mass-energy density parameter Ω_0 (Figure 4). The octopole term ($\ell = 3$) matches WMAP's octopole best when $1.010 < \Omega_0 < 1.014$. Encouragingly, in the subinterval $1.012 < \Omega_0 < 1.014$ the quadrupole ($\ell = 2$) also matches the WMAP value. More encouragingly still, this subinterval agrees well with observations, falling comfortably within WMAP's best fit range of $\Omega_0 = 1.02 \pm 0.02$ (Bennett *et al.*, 2003).

The excellent agreement with WMAP's results is all the more striking because the Poincaré dodecahedral space offers no free parameters in its construction. The Poincaré space is rigid, meaning that geometrical considerations require a completely regular dodecahedron. By contrast, a 3-torus, which is nominally made by gluing opposite faces of a cube but may be freely deformed to any parallelepiped, has six degrees of freedom in its geometrical construction. Furthermore, the Poincaré space is globally homogeneous, meaning that its geometry - and therefore its power spectrum - looks statistically the same to all observers within it. By contrast a typical finite space looks different to observers sitting at different locations.

Confirmation of a positively curved universe ($\Omega_0 > 1$) would require revisions to current theories of inflation, but the jury is still out on how severe those changes would be. Some researchers argue that positive curvature would not disrupt the overall mechanism and effects of inflation, but only limit the factor by which space expands during the inflationary epoch to about a factor of ten (Uzan, Kirchner and Ellis, 2003). Others claim that such models require fine-tuning and are less natural than the infinite flat space model (Linde, 2003).

Having accounted for the weak observed quadrupole, the Poincaré dodecahedral space will face two more experimental tests in the next few years:

- The Cornish-Spergel-Starkman circles–in–the–sky method (Cornish, Spergel and Starkman, 1998) predicts temperature correlations along matching circles in small multiconnected spaces such as this one. When $\Omega_0 \simeq 1.013$ the horizon radius is about 0.38 in units of the curvature radius, while the dodecahedron's inradius and outradius are 0.31 and 0.39, respectively, in the same units; as a result, the volume of the physical space is only 83% the volume of the horizon sphere. In this case the horizon sphere self-intersects in six pairs of circles of angular radius about 35°, making the dodecahedral space a good candidate for circle detection if technical problems (galactic foreground removal, integrated Sachs-Wolfe effect, Doppler effect of plasma motion) can be overcome. Indeed the Poincaré dodecahedral space makes circle searching easier than in the general case, because the six pairs of matching circles must a priori lie in a symmetrical pattern like the faces of a dodecahedron, thus allowing the searcher to slightly relax the noise tolerances without increasing the danger of a false positive.

- The Poincaré dodecahedral space predicts $\Omega_0 \simeq 1.013 > 1$. The upcoming Planck surveyor data (or possibly even the existing WMAP data in conjunction with other data sets) should determine Ω_0 to within 1%. Finding $\Omega_0 < 1.01$ would refute the Poincaré space as a cosmological model, while $\Omega_0 > 1.01$ would provide strong evidence in its favour.

2. Conclusion

Since antiquity humans have wondered whether our universe is finite or infinite. For most of the past two millennia Europeans held the Aristotelian view of the universe as a finite ball with a spherical boundary. The invention of the telescope in 1608 revealed the universe to be much larger than Aristotle had imagined. Thus even though Galileo and Kepler clung to Aristotles model, their successors Bruno, Descartes and especially Newton embraced the idea of infinite space. Nevertheless, some scientists were as uncomfortable with an infinite universe as they were with Aristotle's hypothetical boundary. In 1854 Georg Riemann cut the Gordian knot by proposing the hypersphere as a model of a finite universe with no troublesome boundary. By 1890 Felix Klein discovered the more general concept of a multiconnected space (recall Figure 2) and during the early years of the twentieth century Einstein and others preferred finite universe models. Nevertheless, by the 1930's the vast size of the observable universe had become known and the pendulum swung back towards infinite models. Now, after more than two millennia of speculation, observational data might finally settle this ancient question once and for all.

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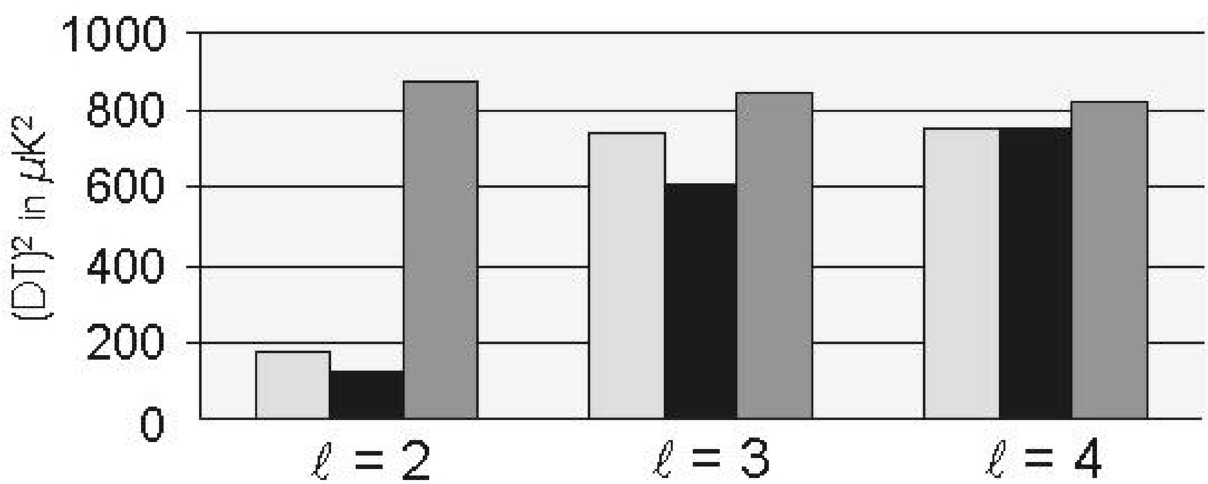

Fig. 1.— Comparison of WMAP power spectrum to Poincaré dodecahedral space and infinite flat universe. At the low end of the power spectrum WMAP's results (drawn in black) match the Poincaré dodecahedral space (light grey) better than they match the expectations for an infinite flat universe (dark grey). Computed for a matter energy density parameter $\Omega_m = 0.28$ and a cosmological constant energy density parameter $\Omega_\Lambda = 0.734$ with Poincaré dodecahedral space data normalised to the $\ell = 4$ term.

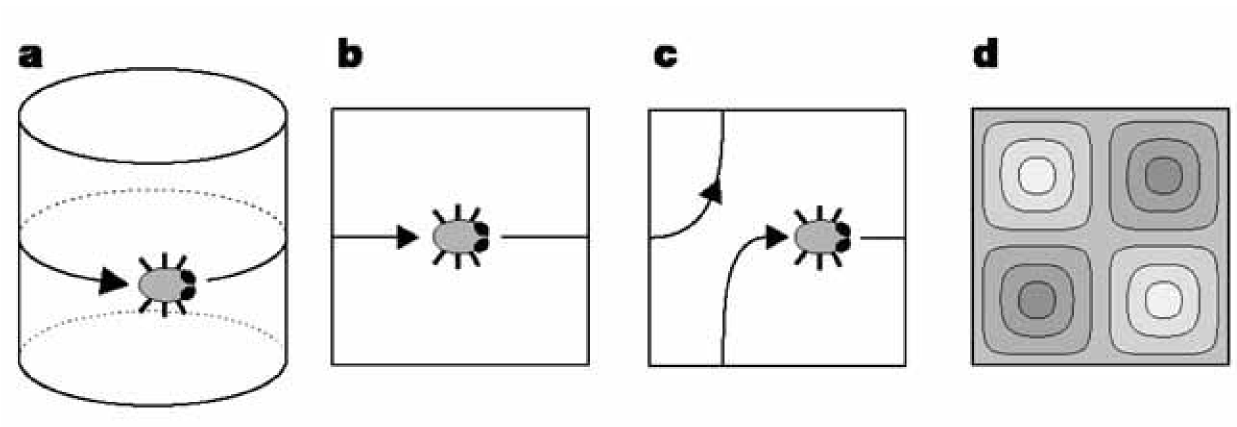

Fig. 2.— Wavelengths of density fluctuations limited by size of finite "wraparound" universe. **a)** A 2-dimensional creature living on the surface of a cylinder travels due east, eventually going all the way around the cylinder and returning to her starting point. **b)** If we cut the cylinder open and flatten it into a square, the creature's path goes out the square's right side and returns from the left side. **c)** A flat torus is like a cylinder, only now the top and bottom sides connect as well as the left and right. **d)** Waves in a torus universe may have wavelengths no longer than the width of the square itself. To construct a multiconnected 3-dimensional space, start with a solid polyhedron (for example a cube) and identify its faces in pairs, so that any object leaving the polyhedron through one face returns from the matching face. Such a multiconnected space supports standing waves whose exact shape depends on both the geometry of the polyhedron and how the faces are identified. Nevertheless, the same principle applies, that the wavelength cannot exceed the size of the polyhedron itself. In particular, the inhabitants of such a space will observe a cut-off in the wavelengths of density fluctuations.

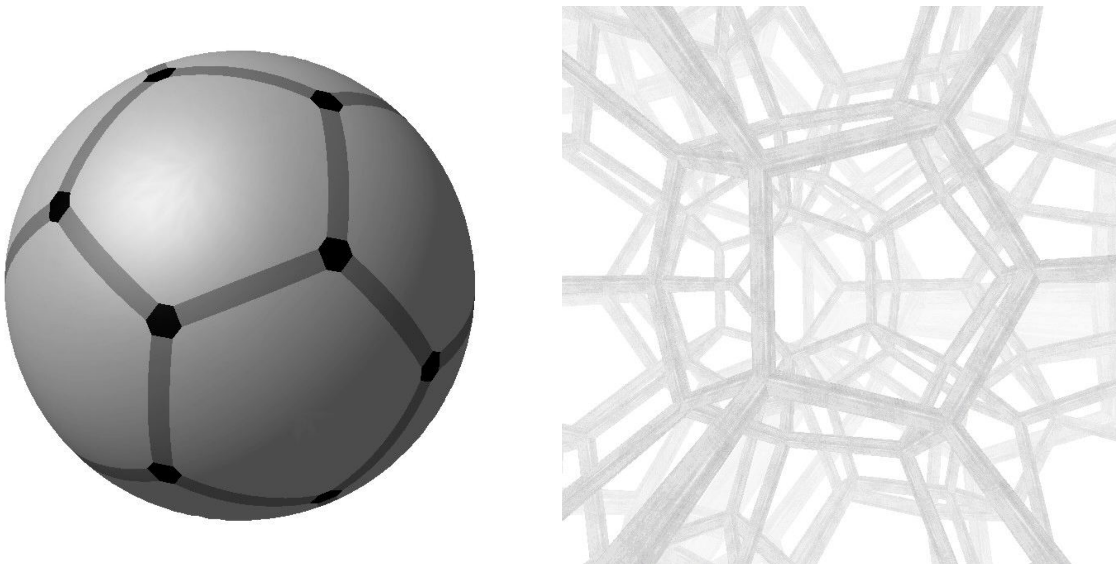

Fig. 3.— Spherical pentagons and dodecahedra fit snugly, unlike their Euclidean counterparts. **a)** Twelve spherical pentagons tile the surface of an ordinary sphere. They fit together snugly because their corner angles are exactly 120°. Note that each spherical pentagon is just a pentagonal piece of a sphere. **b)** One hundred twenty spherical dodecahedra tile the surface of a hypersphere. A hypersphere is the 3-dimensional surface of a 4-dimensional ball. Note that each spherical dodecahedron is just a dodecahedral piece of a hypersphere. The spherical dodecahedra fit together snugly because their edge angles are exactly 120°. In the construction of the Poincaré dodecahedral space the dodecahedron's 30 edges come together in ten groups of three edges each, forcing the dihedral angles to be 120° and requiring a spherical dodecahedron rather than a Euclidean one. Software for visualising spherical dodecahedra and the Poincaré dodecahedral space is available for free download from `www.geometrygames.org/CurvedSpaces`.

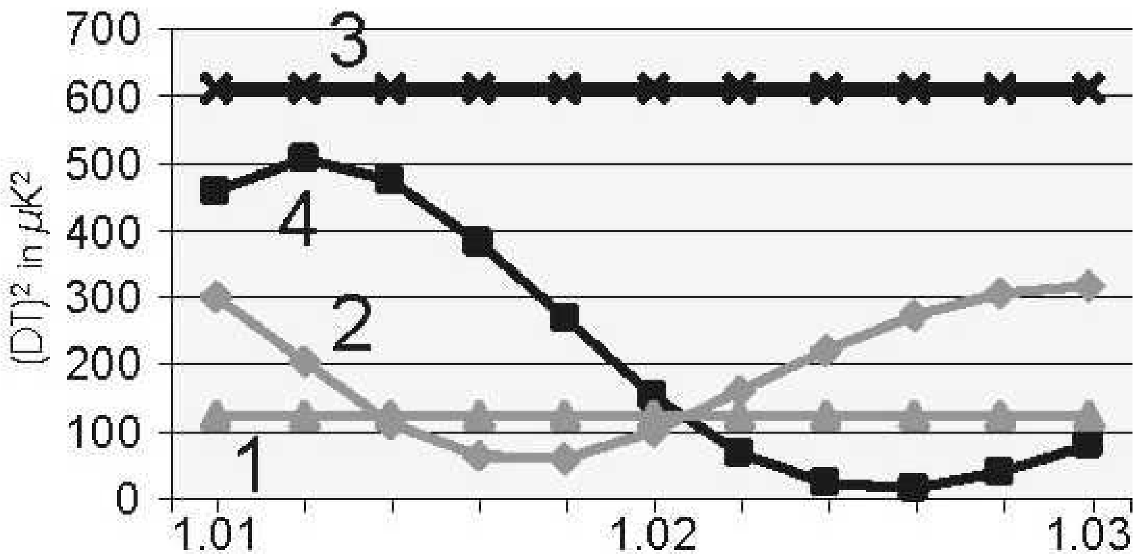

Fig. 4.— Values of the mass-energy density parameter Ω_0 for which the Poincaré dodecahedral space agrees with WMAP's results. The Poincaré dodecahedral space quadrupole (trace 2) and octopole (trace 4) fit the WMAP quadrupole (trace 1) and octopole (trace 3) when $1.012 < \Omega_0 < 1.014$. Larger values of Ω_0 predict an unrealistically weak octopole. To obtain these predicted values we first computed the Poincaré dodecahedral space's eigenmodes using the Ghost Method of Lehoucq *et al.* (2002) with two of the matrix generators computed in Appendix B of Gaussmann *et al.* (2001), and then applied the method of Riazuelo *et al.* (2003), using $\Omega_m = 0.28$ and $\Omega_\Lambda = \Omega_0 - \Omega_m$, to obtain a power spectrum and to simulate sky maps. Numerical limitations restricted our set of 3-dimensional eigenmodes to wavenumbers $k < 30$, which in turn restricted the reliable portion of the power spectrum to $\ell = 2, 3, 4$. We set the overall normalisation factor to match the WMAP data at $\ell = 4$ and then examined the predictions for $\ell = 2, 3$.