Relativistic Arquimedes law for fast moving bodies and the general-relativistic resolution of the "submarine paradox"

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We investigate and solve in the context of General Relativity the apparent paradox which appears when bodies floating in a background fluid are set in relativistic motion. Suppose some macroscopic body, say, a submarine designed to lie just in equilibrium when it rests (totally) immersed in a certain background fluid. The puzzle arises when different observers are asked to describe what is expected to happen when the submarine is given some high velocity parallel to the direction of the fluid surface. On the one hand, according to observers at rest with the fluid, the submarine would contract and, thus, sink as a consequence of the density increase. On the other hand, mariners at rest with the submarine using an analogous reasoning for the fluid elements would reach the opposite conclusion. The general relativistic extension of the Arquimedes law for moving bodies shows that the submarine sinks.

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Suppose a submarine designed to lie just in equilibrium when it rests (totally) immersed in a certain background fluid. The puzzle appears when different observers are asked to describe what is expected to happen when the submarine is given some high velocity parallel to the direction of the fluid surface. On the one hand, according to observers at rest with the fluid, the submarine would contract and sink as a consequence of the density increase. On the other hand, mariners at rest with the submarine using an analogous reasoning for the fluid elements would reach the opposite conclusion. To the best of our knowledge, the first one to discuss this apparent paradox was Supplee [1]. Because his analysis was performed in the context of Special Relativity, assumptions about how the Newtonian gravitational field would transform in different reference frames were unavoidable. In order to set the resolution of this puzzle on more solid bases, a general-relativistic analysis is required. We will adopt hereafter natural units: $c = \hbar = G = k = 1$, and spacetime metric signature $(-,+,+,+)$.

Let us begin writing the line element of the most general spherically symmetric static spacetime as

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2) \ , \quad (1)$$

where $f(r)$ and $g(r)$ are determined by the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$. We will consider the base planet where the experiment will take place as composed of two layers: an interior solid core with total mass M and $r \in [0, R_-]$ $(R_- > 2M)$ and an exterior liquid shell with $r \in (R_-, R_+]$. The gravitational field on the liquid shell will be assumed to be mostly ruled by the solid core, as verified, e.g., on Earth. In this case, the proper acceleration experienced by the static liquid volume elements can be approximated by $(M/r^2)/\sqrt{1-2M/r}$ and, thus, *increases with depth*. This physical feature will be kept as we model the gravitational field on the fluid in which the submarine is immersed, but rather than locating it in the spacetime described by Eq. (1), we will look for a background with planar symmetry. This is necessary in order to avoid the appearance of centrifugal effects which are not part of the submarine paradox. This is accomplished by the Rindler spacetime

$$ds^2 = e^{2\alpha Z}(-dT^2 + dZ^2) + dx^2 + dy^2 \ , \quad (2)$$

where $\alpha = \text{const} > 0$. The liquid layer will be set at $Z \in (Z_-, 0]$, where $Z_- < 0$ and we will assume that $|Z_-| \gg 1/\alpha$ in which case the total proper depth as defined by static observers will be approximately $1/\alpha$. The proper acceleration of the liquid volume elements at some point (T, Z, x, y) is $a_{(1)} = \alpha e^{-\alpha Z}$ and, thus, indeed increases as one moves to the bottom.

Let us assume the submarine to have rectangular shape and to lie initially at rest in the region $x > 0$ at $[Z_\perp, Z_\top] \times [x_\vdash, x_\dashv] \times [y_1, y_2]$. For the sake of simplicity, we will assume the submarine to be thin with respect to the depth $1/\alpha$, i.e. $e^{\alpha Z_\top} - e^{\alpha Z_\perp} \ll 1$. This is not only physically desirable as a way to minimize turbulence and shear effects, but also technically convenient as will be seen further. At $T = 0$ it begins to move along the x-axis towards increasing x values in such a way that eventually its points acquire uniform motion characterized by the 3-velocity $v_0 \equiv dx/dT = \text{const} > 0$. However, in order to keep the submarine uncorrupted, the whole process must be conducted with caution. First of all, we will impose that the 4-velocity $u^\mu_{(\mathrm{s})}$ of the submarine points satisfy the *no-expansion condition*: $\Theta \equiv \nabla_\mu u^\mu_{(\mathrm{s})} = 0$. This can be implemented by the following choice:

$$u^\mu_{(\mathrm{s})} = \frac{\chi^\mu + v(x^\alpha)\zeta^\mu}{|\chi^\mu + v(x^\alpha)\zeta^\mu)|} \ , \quad (3)$$

where $\chi^\mu = (1,0,0,0)$ and $\zeta^\mu = (0,0,1,0)$ are timelike

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and spacelike Killing fields, respectively, and

$$v(x^\alpha) \equiv \frac{dx}{dT} = \begin{cases} 0 & \text{for} \quad T/x < 0 \\ e^{2\alpha Z} T/x & \text{for} \quad 0 \le T/x \le v_0 e^{-2\alpha Z} \\ v_0 & \text{for} \quad T/x > v_0 e^{-2\alpha Z} \end{cases} . \tag{4}$$

Hence, a generic submarine point will have a timelike trajectory in the region $x > 0$, given by $Z = Z_0 = \text{const}$, $y = y_0 = \text{const}$ and

$$x(T) = \begin{cases} x_0 & \text{for} \quad T < 0 \\ \sqrt{x_0^2 + e^{2\alpha Z_0} T^2} & \text{for} \quad 0 \le T \le T_{\rm un}, \\ x_0 \sqrt{1 - v_0^2 e^{-2\alpha Z_0}} + v_0 T & \text{for} \quad T > T_{\rm un} \end{cases} \tag{5}$$

where $T_{\rm un} = x_0 v_0 e^{-2\alpha Z_0} / \sqrt{1 - v_0^2 e^{-2\alpha Z_0}}$ defines the moment after which each submarine point acquires uniform motion with constant 3-velocity v_0 $(0 < v_0 < e^{\alpha Z_0})$.

It should be noticed that the no-expansion requirement is a necessary but not sufficient condition to guaranty that the submarine satisfies the rigid body condition

$$\nabla^{(\mu} u^{\nu)}_{(\rm s)} + a^{(\mu}_{(\rm s)} u^{\nu)}_{(\rm s)} = 0 \;, \tag{6}$$

i.e. that the *proper* distance among the submarine points are kept immutable, where $a^\mu_{(\rm s)} \equiv u^\nu_{(\rm s)} \nabla_\nu u^\mu_{(\rm s)}$. This can be seen by recasting Eq. (6) in the form

$$\sigma_{\mu\nu} + (\Theta/3) P_{\mu\nu} = 0, \tag{7}$$

where $P_{\mu\nu} \equiv g_{\mu\nu} + u^{(\rm s)}_\mu u^{(\rm s)}_\nu$ is the projector operator and

$$\sigma_{\mu\nu} \equiv \left(\nabla_\alpha u^{(\rm s)}_{(\mu} \right) P^\alpha_{\ \ \nu)} - (\Theta/3) P_{\mu\nu}$$

is the shear tensor. If the submarine were infinitely thin $(Z_\perp = Z_\top)$, then $\sigma_{\mu\nu}$ would vanish in addition to Θ and the rigid body equation (7) would be precisely verified. But this is not so because the fact that $Z_\perp \neq Z_\top$ induces shear as the submarine is *in the transition region:* $0 \le T \le T_{\rm un}$.

In order to figure out how this can be minimized, we must first calculate the eigenvalues $\lambda_{(\rm i)}$ (i=1,2,3) and the corresponding (mutually orthogonal) spacelike eigenvectors $w^\mu_{(\rm i)}$ (which also satisfies $w^\mu_{(\rm i)} u^{(\rm s)}_\mu = 0$) associated with the equation $\sigma^\mu{}_\nu w^\nu_{(\rm i)} = \lambda_{(\rm i)} w^\mu_{(\rm i)}$:

$$\lambda_{(1)} = 0, \qquad \lambda_{(2)/(3)} = +/- \sqrt{\sigma^2},$$

$$w^\mu_{(1)} = (0,0,0,1), \quad w^\mu_{(2)/(3)} = (\sigma^0{}_1, +/- \sqrt{\sigma^2}, \sigma^2{}_1, 0) \;,$$

where

$$\sigma^2 \equiv \sigma^{\mu\nu} \sigma_{\mu\nu}/2 = a^2_{(\rm l)} x^2 (x^2 - x_0^2)/x_0^4 \;,$$

$$\sigma^0{}_1 = \alpha e^{-\alpha Z} x (x^2 - x_0^2)/x_0^3, \; \sigma^2{}_1 = \alpha x^2 (x^2 - x_0^2)^{1/2}/x_0^3$$

and we recall that $a_{(\rm l)} = \alpha e^{-\alpha Z}$. Then, by locally choosing a 3-vector basis $e^\mu_{(\rm i)} = w^\mu_{(\rm i)}$ and assuming that $e^\mu_{(\rm i)}$ is orthogonally transported along $u^\mu_{(\rm s)}$, i.e. $[u_{(\rm s)}, e_{(\rm i)}]^\mu = a^{(\rm s)}_\nu e^\nu_{(\rm i)} u^\mu_{(\rm s)}$, one obtains $u^\mu_{(\rm s)} \nabla_\mu |e^\nu_{(\rm i)}| = \lambda_{(\rm i)} |e^\nu_{(\rm i)}|$. Hence, the distortion rate of a sphere inside the submarine along the principal axes $e^\mu_{(\rm i)}$ is given by the corresponding eigenvalues $\lambda_{(\rm i)}$. In our case, no distortion is verified along the y-axis (see $\lambda_{(1)}$ and $w^\mu_{(1)}$) and the distortion which appears in the transition region associated with the Z-axis can be minimized by making $|\lambda_{(2)/(3)}|$ small enough. By using Eq. (5) (at $T = T_{\rm un}$), one obtains

$$|\lambda_{(2)}| = |\lambda_{(3)}| \le a_{(\rm l)} v_0 e^{-\alpha Z_\perp} / (1 - v_0^2 e^{-2\alpha Z_\perp}) \;.$$

Thus one can minimize shear effects in the submarine either (i) by making the final velocity to be moderate $(v_0 \ll e^{\alpha Z_\perp})$, (ii) by setting it in a small-acceleration region [in comparison to the inverse of the submarine Z-proper size: $a_{(\rm l)} \ll \alpha/(e^{\alpha Z_\top} - e^{\alpha Z_\perp})$], or, as considered here, (iii) by designing the submarine thin enough $(e^{\alpha Z_\top} - e^{\alpha Z_\perp} \ll 1)$.

After the transition region, all the submarine points will follow isometry curves associated with the timelike Killing field $\eta^\mu = \chi^\mu + v_0 \zeta^\mu$. It is easy to check by using

$$a^\mu_{(\rm s)} = (\nabla^\mu \eta)/\eta = (0, \alpha e^{-2\alpha Z}/(1 - v_0^2 e^{-2\alpha Z}), 0, 0) \;, \tag{8}$$

where $\eta \equiv |\eta^\mu| = e^{\alpha Z}(1 - v_0^2 e^{-2\alpha Z})^{1/2}$ that the rigid body equation is fully verified in this stationary region: $T > T_{\rm un}$. It is interesting to notice that although mariners aboard will not perceive any significant change in the submarine's form, observers at rest with the fluid will witness a relevant contraction in the x-axis direction as a function of Z (and v_0); indeed, more at the top than at the bottom (see Fig. 1).

Now, let us suppose that the liquid layer in which the submarine is immersed is a perfect fluid characterized by the energy-momentum tensor

$$T^{\mu\nu} = \rho_{(\rm l)} u^\mu_{(\rm l)} u^\nu_{(\rm l)} + P_{(\rm l)} (g^{\mu\nu} + u^\mu_{(\rm l)} u^\nu_{(\rm l)}) \;,$$

where $u^\mu_{(\rm l)} = \chi^\mu/\chi$ with $\chi = |\chi^\mu| = e^{\alpha Z}$, and $\rho_{(\rm l)}$ and $P_{(\rm l)}$ are the fluid's proper energy density and pressure, respectively. From $\nabla_\mu T^{\mu\nu} = 0$, we obtain

$$\nabla^\mu P_{(\rm l)} = -(\rho_{(\rm l)} + P_{(\rm l)}) a^\mu_{(\rm l)} \;, \tag{9}$$

where $a^\mu_{(\rm l)} = (0, \alpha e^{-2\alpha Z}, 0, 0)$. For later convenience, we cast Eq. (9) in the form

$$\rho_{(\rm l)} d\chi/dl + d(\chi P_{(\rm l)})/dl = 0 \;, \tag{10}$$

where we have used that $a^\mu_{(\rm l)} = (\nabla^\mu \chi)/\chi$ and dl is the differential proper distance in the Z-axis direction.

The *proper* hydrostatic pressures at the bottom $P_\perp$ and on the top $P_\top$ of the submarine will be given by

$$P_{\perp/\top} \equiv T_{\mu\nu} N^\mu_{\perp/\top} N^\nu_{\perp/\top} = P_{(\rm l)}|_{Z = Z_{\perp/\top}} \;, \tag{11}$$

where $N^{\mu}_{\perp/\top} = (0,1,0,0)e^{-\alpha Z_{\perp/\top}}$ are unit vectors orthogonal to the submarine's 4-velocity (and to the top and bottom surfaces). Thus, the hydrostatic scalar forces on the top and at the bottom of the submarine are

$$F_{\perp/\top} = +/- AP_{\perp/\top} = +/- AP_{(1)}|_{Z=Z_{\perp/\top}} ,$$

where A is the corresponding proper area.

In order to combine $F_{\perp}$ and $F_{\top}$ properly, we must transmit them to a common holding point. Let us assume that the forces are transmitted through a lattice of ideal cables and rods to some arbitrary inner point $\mathcal{O} \equiv (Z_{\mathcal{O}}, x_{\mathcal{O}}, y_{\mathcal{O}})$ inside the submarine, where its mass is also concentrated. Ideal cables and rods are those ones which transmit pressure through $\nabla_{\mu} T^{\mu\nu} = 0$ and have negligible energy densities. As a consequence of our thin-submarine assumption, our final answer will be mostly insensitive to the choice of $\mathcal{O}$. $F_{\perp/\top}$ are related to the transmitted forces $F^{\mathcal{O}}_{\perp/\top}$ at $\mathcal{O}$ by $F^{\mathcal{O}}_{\perp/\top} = [\eta(Z_{\perp/\top})/\eta(Z_{\mathcal{O}})]F_{\perp/\top}$. Hence, the Arquimedes law induces the following scalar force (along the Z-axis) at $\mathcal{O}$

$$F^{\mathcal{O}}_{\rm A} = F^{\mathcal{O}}_{\perp} + F^{\mathcal{O}}_{\top} = - \left. \frac{V}{\eta(Z)} \frac{d(\eta(Z)P_{(1)})}{dl} \right|_{Z=Z_{\mathcal{O}}} , \qquad (12)$$

where V is the submarine's proper volume and we have assumed that $d(\eta(Z)P_{(1)})/dl$ does not vary much along the submarine so that we can neglect higher derivatives. This is natural in light of our thin-submarine assumption.

In addition to $F^{\mathcal{O}}_{\rm A}$, we must consider the force (along the Z-axis) associated with the gravitational field:

$$\begin{aligned} F^{\mathcal{O}}_{\rm g} &= -m a^{\mu}_{(\rm s)} N_{\mu}|_{Z=Z_{\mathcal{O}}} \\ &= -mN^{\mu}(\nabla_{\mu}\eta)/\eta|_{Z=Z_{\mathcal{O}}} \\ &= -\,(m/\eta(Z))(d\eta(Z)/dl)|_{Z=Z_{\mathcal{O}}} , \qquad (13) \end{aligned}$$

where $a^{\mu}_{(\rm s)}|_{Z=Z_{\mathcal{O}}}$ is obtained from Eq. (8), m is the submarine mass and $N_{\mu}|_{Z=Z_{\mathcal{O}}} = (0,1,0,0)e^{\alpha Z_{\mathcal{O}}}$.

Now, by adding up Eqs. (12) and (13) we obtain the total force on the submarine as

$$F^{\mathcal{O}}_{\rm tot} = - \left[\frac{m}{\eta(Z)} \frac{d\eta(Z)}{dl} + \frac{V}{\eta(Z)} \frac{d(\eta(Z)P_{(1)})}{dl} \right]_{Z=Z_{\mathcal{O}}} . \quad (14)$$

In order to fix the submarine's mass, we give to it just the necessary ballast to keep it in hydrostatic equilibrium when it lies at rest completely immersed. This means that we must impose $F^{\mathcal{O}}_{\rm tot}|_{v_0=0} = 0$. Now, by recalling that $\eta \stackrel{v_0\to 0}{\longrightarrow} \chi$ and using Eq. (10), we reach the conclusion [2] that the equilibrium condition above implies that the submarine must be designed such that its mass-to-volume ratio obey the simple relation $m/V = \rho_{(1)}$. Then, by using this and Eq. (10) in Eq. (14), it is not difficult to write the total proper force on the moving submarine as

$$\begin{aligned} F^{\mathcal{O}}_{\rm tot} &= -V(\rho_{(1)} + P_{(1)}) \left(\frac{1}{\eta}\frac{d\eta}{dl} - \frac{1}{\chi}\frac{d\chi}{dl} \right) \Bigg|_{Z=Z_{\mathcal{O}}} \\ &= -V(\rho_{(1)} + P_{(1)}) N^{\mu} \left(\frac{\nabla_{\mu}\eta}{\eta} - \frac{\nabla_{\mu}\chi}{\chi} \right) \Bigg|_{Z=Z_{\mathcal{O}}} \end{aligned}$$

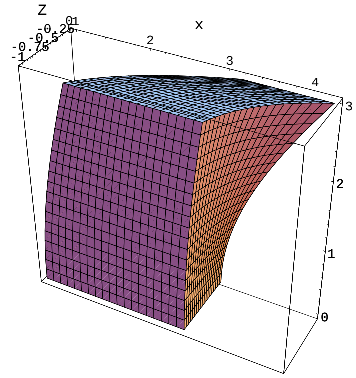

FIG. 1: The time evolution of a $y = $ const section is plotted (using $\alpha = 1$). At $T = 0$, the submarine is at rest and all the observers agree about its rectangular shape. As its velocity increases, however, the submarine contracts as a function of Z according to the observers at rest with the fluid [more on the top than at the bottom (see slices $T = $ const > 0)], although mariners aboard in will detect no relevant change in shape.

and, thus,

$$F^{\mathcal{O}}_{\rm tot} = \left. \frac{-V(\rho_{(1)} + P_{(1)})a_{(1)}v_0^2 e^{-2\alpha Z}}{1 - v_0^2 e^{-2\alpha Z}} \right|_{Z=Z_{\mathcal{O}}} , \qquad (15)$$

where we recall that $a_{(1)} = \alpha e^{-\alpha Z}$. Clearly, for $v_0 = 0$ we have $F^{\mathcal{O}}_{\rm tot} = 0$, as it should be, but for $v_0 \neq 0$ we have $F^{\mathcal{O}}_{\rm tot} < 0$ and, thus, *we conclude that a net force downwards is exerted on the submarine.*

In order to make contact of this result with the one obtained through Special Relativity, let us begin by assuming $\rho_{(1)} = \rho_0 = $ const, in which case we can easily solve Eq. (9): $P_{(1)} = \rho_0(e^{-\alpha Z} - 1)$. By letting this in Eq. (15), we obtain

$$F^{\mathcal{O}}_{\rm tot} = - \left. \frac{m\alpha v_0^2 e^{-4\alpha Z}}{1 - v_0^2 e^{-2\alpha Z}} \right|_{Z=Z_{\mathcal{O}}} . \qquad (16)$$

Now, let us assume that the submarine is close to the surface, i.e. at $Z \approx 0$, in which case the line element (2) reduces to the usual line element form of the Minkowski space with (T, Z, x, y) playing the role of the Cartesian coordinates. As a consequence, Eq. (16) reduces to

$$F^{\mathcal{O}}_{\rm tot} \approx -mg\gamma(\gamma - 1/\gamma)|_{Z\approx 0} , \qquad (17)$$

where $\gamma \equiv 1/\sqrt{1 - v_0^2}$ and we have assumed that the gravitational field is small enough such that we can identify the proper acceleration on the liquid volume elements $a_{(1)} = \alpha e^{-\alpha Z} \stackrel{Z\to 0}{\longrightarrow} \alpha$ with the *Newtonian* gravity acceleration g. Notice that the first and second terms in Eq. (17) can be associated with the proper gravitational and buoyancy forces, respectively. Finally, by evoking Special Relativity to transform the force from the proper frame of the submarine (17) to the one at rest with the fluid, we reobtain Supplee's formula [1]:

$$F_{\rm tot} = -mg(\gamma - 1/\gamma) .$$

Thus according to observers at rest with the fluid, the gravitational field on the moving submarine increases effectively by a γ factor as a consequence of the blue-shift on the submarine's energy and the buoyancy force decreases by the same factor because of the volume contraction. The apparently contradictory conclusion reached in the submarine rest frame by the mariners, who would witness a density increase of the liquid volume elements is resolved by recalling that the gravitational field is not going to "appear" the same to them as to the observers at rest with the fluid. This is naturally taken into account in the General-Relativistic approach (and turned out to be the missing ingredient which raised the paradox). This can be seen from Eq. (13) by casting it in the form $F_{\rm g}^{\mathcal{O}} = -m\alpha e^{-\alpha Z_{\mathcal{O}}}/(1 - v_0^2 e^{-2\alpha Z_{\mathcal{O}}})$. Hence, according to mariners aboard, the effective gravitational force will be larger when the submarine is moving than when it is at rest by a factor $(1 - v_0^2 e^{-2\alpha Z_{\mathcal{O}}})^{-1} > 1$, pushing it downwards.

The Theory of Relativity is close to commemorate its first centennial. This is quite remarkable that it has not lost the gift of surprising us so far. This is definitely a privilege of few elders.

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